On some Applications of Laplace-Transformation to the Heat Transfer Problems. (Report 5)

Ryotaro IZUMI

Mechanical Engineering Dep., Yamagata Technical Colledge.

Introdution.

In this paper, we treat the case of natural heat convection inside the vertical tube of infinite length and for the case of constant wall temperature we have already discussed in Report 1.

Heat quantity is applied to the vertical wall, as shown in Fig. 1, the temerature and velocity are increased gradually with the time, and this apparatus is widely used for a drying tower and heating of a fluid inside the vertical tube.

1. Fundamental Equation and Initial and Boundary Conditions.

Consider the one-dimensional upward flow, the temperature and velocity distributions $\vartheta(r,t)$, $u(r,t)$ are deduced from the following equations

$$\partial\vartheta/\partial t=\kappa\left(\frac{\partial^2\vartheta}{\partial r^2}+\frac{1}{r}\frac{\partial\vartheta}{\partial r}\right) \qquad (1)$$

and,

$$\partial u/\partial t=\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+g\beta\vartheta \qquad (2)$$

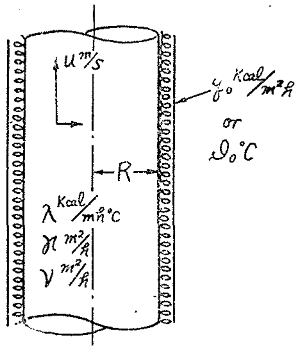

Fig. 1

where, κ, ν, g and β are thermal diffusivity, kinematic viscosity, acceleration of gravity and thermal expansion coefficient respectively.

Ininial and boundary conditions are as follows;

$$t=0 : \quad \vartheta=0 \text{ and } u=0 \qquad (3)$$

$$r=0 : \quad \vartheta, u= \text{ finite} \qquad (4)$$

$$r=R : \quad -\lambda\,\partial\vartheta/\partial r = q_0 = \text{constant heat quantity and } u=0 \qquad (5)$$

where, λ is the thermal conductivity of fluid.

2. Theoretical Solution.

To solve there equations under the conditions (3), (4) and (5), we introduce an operator p refer to time t.

Using a Laplace-Transformation, the transformed equations of (1) and (2) which satisfy the condition (3) are

$$\frac{d^2\bar{\vartheta}}{dr^2}+\frac{1}{r}\frac{d\bar{\vartheta}}{dr}-\frac{p}{\kappa}\bar{\vartheta}=0 \tag{6}$$

and,

$$\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}-\frac{p}{\nu}\bar{u}+g\beta\bar{\vartheta}=0 \tag{7}$$

From the above equation we have the solutions of transformed temperature $\bar{\vartheta}\ (p,r)$ and velocity $\bar{u}\ (p,r)$.

$$\bar{\vartheta}(p,r)=\frac{q_0 I_0(\sqrt{p/\kappa}\,r)}{\lambda\sqrt{p/\kappa}\ p\ I_1(\sqrt{p/\kappa}\,R)} \tag{8}$$

and,

$$\bar{u}(p,r)=\frac{g\beta q_0\sqrt{\kappa}\,I_0(p/\kappa\,R)\ I_0(\sqrt{p/\nu}\,R)}{\lambda\nu\left(\frac{1}{\kappa}-\frac{1}{\nu}\right)\sqrt{p}\,p^2 I_0(\sqrt{p/\nu}\,R)\ I_1(\sqrt{p/\kappa}\,R)}$$

$$-\frac{g\beta q_0\sqrt{\kappa}\ \ I_0(\sqrt{p/\kappa}\,r)}{\lambda\nu\left(\frac{1}{\kappa}-\frac{1}{\nu}\right)\sqrt{p}\,p^2\ I_1(\sqrt{p/\kappa}\,R)} \tag{9}$$

where, I_0, and I_1 are the modified Bessel Functions of the first kind of order zero and one.

To find the inverse transformations of (8) and (9), in the denominator, as the singular points are the points coresponding to $p\sqrt{p}=0$ and $J_1(i\sqrt{p/\kappa}\,R)=0$, these solutions are as follows respectively.

$$\frac{I_0(\sqrt{p/\kappa}\,r)}{p\sqrt{p}\ I_0(p/\kappa\,R)}=\frac{1}{p\sqrt{p}}\left[\left(\frac{2^2\kappa}{R^2\sqrt{p}}+\frac{\sqrt{p}\,r^2}{R^2}-\frac{\sqrt{p}}{2}+\cdots\cdots\right)\frac{R}{2\sqrt{\kappa}}\right]$$

$$=\frac{R}{2\sqrt{\kappa}}\left(\frac{4\kappa}{p^2R^2}+\frac{r^2}{p\,R_2}-\frac{1}{2p}\right)\overset{p}{\bullet}\!\!-\!\!-\!\!\overset{t}{\circ}\ \frac{R}{2\sqrt{\kappa}}\left(\frac{4\kappa t}{R^2}+\frac{r^2}{R^2}-\frac{1}{2}\right)$$

and,

$$\frac{I_0(\sqrt{p/\kappa}\,r)}{p\sqrt{p}\ I_1(\sqrt{p/\kappa}\,R)}=\frac{J_0(i\sqrt{p/\kappa}\,r}{i^{-1}J_1(i\sqrt{p/\kappa}\,R)\ p\sqrt{p}}$$

$$\overset{p}{\bullet}\!\!-\!\!-\!\!\overset{t}{\circ}\ -2\frac{p}{\sqrt{\kappa}}\sum_{n=0}^{\infty}\frac{J_0(\alpha_n r/R)}{\alpha_n^2\ J_0(\alpha_n)}e^{-\alpha_n^2\frac{\kappa t}{R^2}}$$

Thereforeme, we find the solutions of non-dimensional form as

$$\vartheta\Big/\frac{q_0R}{\lambda}=\frac{\varepsilon^2}{2}+2F_0-\frac{1}{4}-2\sum_{n=0}^{\infty}\frac{J_0(\alpha_n)}{\alpha_n^2 J_0(\alpha_n)}e^{-\alpha_n^2 F_0} \tag{10}$$

where, $\varepsilon=$ r/R, F_0=Fourier's number= $\kappa t/R^2$ and $\alpha_{1,\,2,\,3\,\cdots\cdots\,n}$ are the succesive roots of $J_1(\alpha_n)=0$.

In equation (9), we consider the convolutions, i. e.,

$$L^{-1}\Big[\Big(\frac{1}{p}\Big)\Big(\frac{I_0(\sqrt{p/\kappa}\,r)}{p\sqrt{p}\cdot I_1(\sqrt{p/\kappa}\,R)}\Big)\Big]=1*\Big[\frac{p}{\sqrt{\kappa}}\Big(2\frac{\kappa t}{R^2}+\frac{r^2}{R^2}-\frac{1}{4}\Big)-2\frac{R}{\sqrt{\kappa}}\sum_{n=0}^{\infty}\frac{J_0(\alpha_n\frac{r}{R})}{\alpha_n^2 J_0(\alpha_n)}e^{-\frac{\kappa\alpha_n^2 t}{R^2}}\Big]$$

and,

$$L^{-1}\Big[\Big(\frac{I_0(\sqrt{p/\nu}\,r)}{p\,I_0(\sqrt{p/\nu}\,R)}\Big)\cdot\Big(\frac{I_0(\sqrt{p/\kappa}\,R)}{p\sqrt{p}\;I_1(\sqrt{p/\kappa}\,R)}\Big)\Big]=\Big(1-2\sum_{n=0}^{\infty}\frac{J_0(\beta_n r/R)}{J_1(\beta_n)\beta_n}e^{-\frac{\nu t}{R^2}\beta_n^2}\Big]*\Big[\frac{R}{\sqrt{\kappa}}\Big(2\frac{\kappa t}{R^2}+\frac{1}{2}-\frac{1}{4}\Big)-2\frac{R}{\sqrt{\kappa}}\sum_{n=0}^{\infty}\frac{e^{-\frac{\kappa\alpha_n^2 t}{R^2}}}{\alpha_n^2}\Big]$$

Finally, we have the velocity distribution as follows;

$$u\Big/\frac{g\beta q_0R^3}{\lambda\nu(1-pr)}=2\sum_{n=0}^{\infty}\frac{1-\dfrac{J_0(\alpha_n\varepsilon)}{J_0(\alpha_n)}}{\alpha_n^4}\,pr\,(1-e^{-\alpha_n^2F_0})-\frac{1-\varepsilon^2}{2}F_0\,pr$$
$$+\sum_{n=0}^{\infty}\frac{4J_0(\beta_n\varepsilon)}{\beta_n^4J_1(\beta_n)pr}\,e^{-\beta_n^2F_0pr}\Big[1-e^{-\beta_n^2F_0pr}(1+\beta_n^2F_0pr)$$
$$+\sum_{n=0}^{\infty}\frac{J_0(\beta_n\varepsilon)}{2\beta_n^2J_1(\beta_n)}(1-e^{-\beta_n^2F_0pr})-\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{4J_0(\beta_n\varepsilon)\,J_0(\alpha_n\varepsilon)}{\beta_nJ_1(\beta_n)\;\alpha_n^2J_0(\alpha_n)}$$
$$\times\frac{pr\;e^{-\beta_n^2F_0pr}}{pr\,\beta_n^2+\alpha_n^2}(1-e^{-\beta_n^2F_0pr-\alpha_n^2F_0})\tag{11}$$

where, $\beta_{1,2,\cdots\cdots n}$ are the succesive roots of $J_0(\beta_n)=0$.

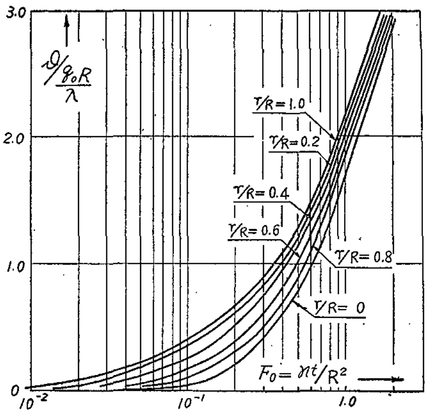

Fig. 2

3. Graphical Representation of the Results.

Fig 2. and 3 show the temperature and velocity distributions $\vartheta/\frac{q_0R}{\lambda}$ and $n/\frac{g\beta\,q_0R^3}{\lambda\nu(1-pr)}$ calculated from equations (10) and (11) respectively.

In each cases these values are represented versus to Fourier's number $F_0=\kappa t/R^2$ and graphed refer to the positions r/R=0, 0.2, 0.4, 0.6, 0.8 and 1.0.

We experiment the velocity distribution

using the tower with 6,000 cm length and 28 cm diameter and in the case of $q_0=2\ KW/m^2$, $\vartheta_0=100°C$ the velocity after 30 min. elapsed is $u=2\sim3$ m/s by measuring with the hot-wire-anemometer.

From the graph we obtain the value $u/\frac{g\beta q_0 R^3}{\lambda\nu(1-pr)}=0.088$ when $F_0=0.36$ and thus u is 2.34 m/s, where we takes the value of $\lambda=0.0264$ Kcal/mh°C, $\nu=0.237$ cm²/s and $\kappa=0.3281$ cm²/s.

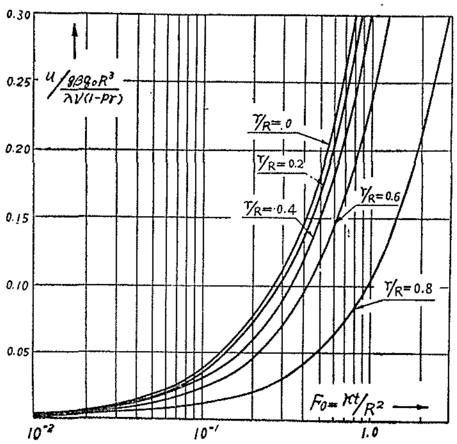

Fig. 3

For the case of constant wall temperature $\vartheta_0°$ C, from the equation (18) in the previous paper, the temperature and velocity distributions are represented in Fig.4 and 5, and these mean values in Fig. 6 versus to Fourier's number. $F_0=\kappa t/R^2$.

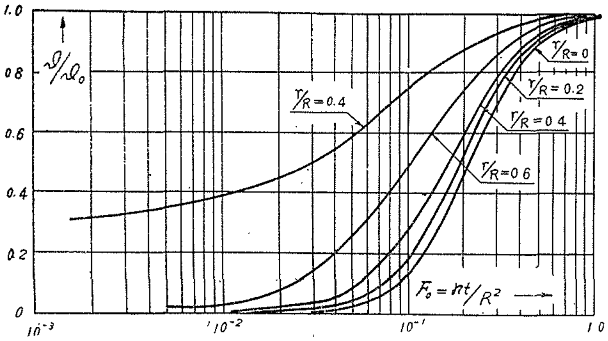

Fig. 4

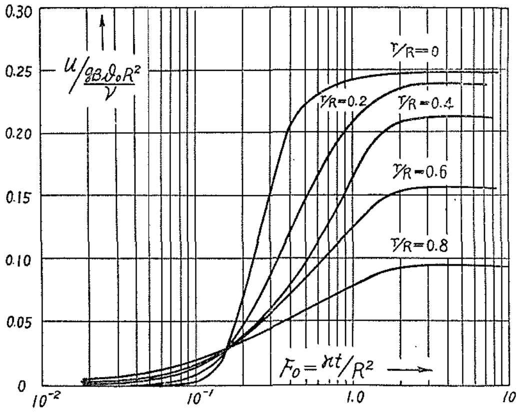

Fig. 5

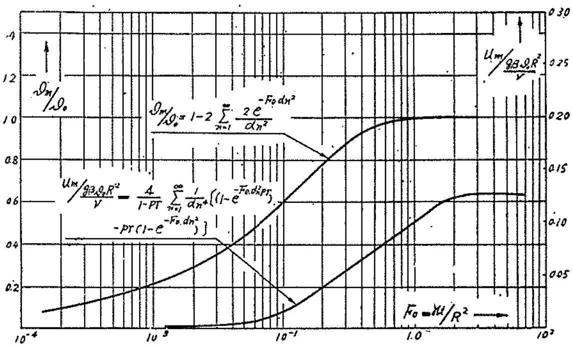

Fig. 6

Reference

1). R. IZUMI : Bull. of Yamagata Univ., (Emgineering), Vol.2, No.1, Nov. 1952,p. 99.

伝熱問題におけるラプラス変換の應用例（その5）

泉 亮太郎

工業短期大学部 機械科

第5報として今回は無限長の円筒壁内の熱対流を取上げ，周壁に熱量が与えられた場合の温度分布および速度分布を求めた。壁面温度が一定温度の場合については 第1報 で求めておいたので，結果の図表のみを掲げた。

この問題は，「円筒壁面内の自然対流について」と題し昭和31年10月14日応用力学連合講演会（京都）において発表した。